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AN OVERVIEW OF TRANSFER FUNCTION FORMALISM FOR NONLINEAR SYSTEMS

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Abstract

Although the Laplace and *Z* transforms of nonlinear differential and respectively difference equations are not defined a transfer function formalism for nonlinear continuous-, discrete-time and time-delay systems was developed recently. Such a formalism shows many properties we expect from transfer functions and in this paper we provide its overview and discuss some basic properties.

Keywords: nonlinear continuous-time systems, nonlinear discrete-time systems, nonlinear time-delay systems, transfer functions, algebraic approach, non-commutative rings

1 INTRODUCTION

Although the Laplace and *Z* transforms are not defined when control systems are nonlinear the transfer function formalism was developed recently. For continuous-time case it was given in [2], [5] and [14], for discrete-time case in [3] and [10] and for time-delay systems in [7]. Such a formalism is, in principle, similar to the linear theory, except that the polynomial description relates now the differentials of system inputs and outputs, and the resulting polynomial ring is non-commutative. In what follows we provide an overview of this formalism and discuss some basic properties.

2 ALGEBRAIC FRAMEWOK

Modern development in nonlinear control theory is related mainly to the use of differential algebraic methods. Algebraic formalism of differential one-forms, originally developed to handle nonlinear continuous-time systems [1], was later extended to the discrete-time case [4] and recently also to the case of nonlinear systems with time delays [6]. It considers so-called generic properties and from such a point of view nonlinear control systems, forming the scope of our attention, are objects of the form

$$\begin{array}{lll} \dot{x}(t)=f(x(t),u(t)) & x(t+1)=f(x(t),u(t)) & \dot{x}(t)=f(\{x(t-i),u(t-j);i,j\geq 0\}) \\ y(t)=g(x(t),u(t)) & y(t)=g(x(t),u(t)) & y(t)=g(\{x(t-i),u(t-j);i,j\geq 0\}) \end{array} \quad (1)$$

where the entries of f and g are meromorphic functions, which we think of as elements of the quotient field of the ring of analytic functions, and $x(t)\in R^n$, $u(t)\in R^m$ and $y(t)\in R^p$ denote the state, the input and the output to the system, respectively.

Let $\mathcal{K}$ denote the field of meromorphic functions of the independent variables; that is, $\{x(t), u^{(k)}(t); k \geq 0\}$ for continuous-time case, $\{x(t), u(t+k); k \geq 0\}$ for discrete-time case and $\{x(t-i), u^{(k)}(t-j); i, j, k \geq 0\}$ for time-delay case.

Let ε be the formal vector space over $\mathcal{K}$ given by

$$\varepsilon = \operatorname{span}_{\mathcal{K}} \{\mathrm{d}\xi; \xi \in \mathcal{K}\} \tag{2}$$

Both, $\mathcal{K}$ and ε, can be endowed with the structure given by the equations (1) by defining differential, shift and, respectively, time-delay operators. For more details and additional references see [1], [4] and [6].

Remark 1. Note that in case of rational (or even algebraic) functions the elements of the vector space ε can be understood as Kähler differentials [11] having been widely used to study nonlinear control systems, see for instance [12] and references therein. However, if one tends to include functions like $\exp(\cdot)$ or $\ln(\cdot)$, as for instance in Example 6, it is necessary to employ the differential one-forms as introduced in [1].

2.1 Skew polynomials

The next step is to extend this algebraic point of view by introducing certain non-commutative rings of skew polynomials defined over the field of meromorphic functions $\mathcal{K}$. Such polynomials play a role of differential, shift and time-delay operators, see [2], [3], [5], [7] [10], [13] and [14].

Continuous-time case. The differential field $\mathcal{K}$ and the derivative operator $\mathrm{d}/\mathrm{d}t$ induce the left skew polynomial ring $\mathcal{K}[s]$ of polynomials in s over $\mathcal{K}$ with usual addition and the non-commutative multiplication given by the commutation rule

$$sa(t) = a(t)s + \dot{a}(t) \tag{3}$$

for any $a(t) \in \mathcal{K}$.

The commutation rule (3) actually represents the rule for differentiating. Polynomials from ring $\mathcal{K}[s]$ thus represent differential operators and act over the vector space ε in the following way

$$\left(\sum_{i=0}^{k} a_i s^{i}\right) v(t) = \sum_{i=0}^{k} a_i v^{(i)}(t) \tag{4}$$

for any $v(t) \in \varepsilon$.

Discrete-time case. The difference field $\mathcal{K}$ and the forward shift operator δ induce the left skew polynomial ring $\mathcal{K}[\delta]$ of polynomials in δ over $\mathcal{K}$ with usual addition and the non-commutative multiplication given by the commutation rule

$$\delta a(t) = a(t+1)\delta \tag{5}$$

for any $a(t) \in \mathcal{K}$.

The commutation rule (5) actually represents the rule for forward shifting. Polynomials from ring $\mathcal{K}[\delta]$ thus represent shift operators and act over the vector space ε in the following way

$$\left(\sum_{i=0}^{k} a_i \delta^{i}\right) v(t) = \sum_{i=0}^{k} a_i v(t+i) \tag{6}$$

for any $v(t) \in \varepsilon$.

Time-delay case. Finally, the differential field $\mathcal{K}$ and the derivative operator $\mathrm{d}/\mathrm{d}t$ and the time-delay operator δ induce the left skew polynomial ring $\mathcal{K}[\delta, s]$ of polynomials in δ and s over $\mathcal{K}$ with usual addition and the non-commutative multiplication given by the commutation rules

$$\begin{aligned} sa(t) &= a(t)s + \dot{a}(t) \\ \delta a(t) &= a(t-1)\delta \\ \delta s &= s\delta \end{aligned} \tag{7}$$

for any $a(t) \in \mathcal{K}$.

The commutation rules (7) actually represent the rules for differentiating and, respectively, backward shifting (time-delaying). Polynomials from ring $\mathcal{K}[\delta, s]$ thus represent differential time-delay operators and act over the vector space ε in the following way

$$\left(\sum_{i,j=0}^{k} a_{ij}\delta^{\mathrm{i}} s^{\mathrm{j}}\right) v(t) = \sum_{i,j=0}^{k} a_{ij} v^{(j)}(t-i) \tag{8}$$

for any $v(t) \in \varepsilon$.

Note that while in the discrete-time case δ stands for forward shift operator, takes t to $t + 1$, in the time-delay systems it is assumed to be the time-delay operator and takes t to $t - 1$. Whether we think of δ as forward shift or, respectively, time-delay operator will be clear from the context or explicitly stated otherwise.

All skew polynomial rings satisfy the so-called Ore condition and can be thus embedded to the non-commutative quotient fields, see [2], [3], [5], [7], [10], [15] and [16] for more details.

Addition is defined by reducing two quotients to the same denominator

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{\beta_2 a_1 + \beta_1 a_2}{\beta_2 b_1} \tag{9}$$

where $\beta_2 b_1 = \beta_1 b_2$ by Ore condition. Multiplication is defined by

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{\alpha_1 a_2}{\beta_2 b_1} \tag{10}$$

where $\beta_2 a_1 = \alpha_1 b_2$ again by Ore condition.

Of course, due to the non-commutative multiplications (3), (5) and (7) the addition and multiplication of quotients of skew polynomials differ from the usual rules.

2.2 Transfer functions

Once the fraction of two skew polynomials is defined we can introduce transfer function of the nonlinear system of the form (1) as a quotient F of skew polynomials such that $\mathrm{d}y(t) = F\mathrm{d}u(t)$.

Example 1. Consider the nonlinear continuous-time system $\dot{y}(t) = y(t)u(t)$. Then

$$\begin{aligned} \mathrm{d}\dot{y}(t) &= u(t)\mathrm{d}y(t) + y(t)\mathrm{d}u(t) \\ s\mathrm{d}y(t) &= u(t)\mathrm{d}y(t) + y(t)\mathrm{d}u(t) \end{aligned}$$

and transfer function

$$F(s) = \frac{y(t)}{s - u(t)}$$

Example 2. Consider the nonlinear discrete-time system $y(t+2)=y(t)u(t+1)+u(t)$. Then

$$\mathrm{d}y(t+2)=u(t+1)\mathrm{d}y(t)+y(t)\mathrm{d}u(t+1)+\mathrm{d}u(t)$$
$$\delta^2\mathrm{d}y(t)=u(t+1)\mathrm{d}y(t)+y(t)\delta\mathrm{d}u(t)+\mathrm{d}u(t)$$

and transfer function

$$F(\delta)=\frac{y(t)\delta+1}{\delta^2-u(t+1)}$$

Example 3. Consider the nonlinear time-delay system $\ddot{y}(t)=\dot{y}(t-1)u(t-1)$. Then

$$\mathrm{d}\ddot{y}(t)=u(t-1)\mathrm{d}\dot{y}(t-1)+\dot{y}(t-1)\mathrm{d}u(t-1)$$
$$s^2\mathrm{d}y(t)=u(t-1)\delta s\mathrm{d}y(t)+\dot{y}(t-1)\delta\mathrm{d}u(t)$$

and transfer function

$$F(\delta,s)=\frac{\dot{y}(t-1)\delta}{s^2-u(t-1)\delta s}$$

Note again that while in discrete-time case δ stands for forward shift operator, takes t to $t+1$, in time-delay systems it is assumed to be the time-delay operator and takes t to $t-1$.

Of course, transfer functions can be computed also from the system equations (1). This is demonstrated in the following example.

Example 4. Consider the the nonlinear time-delay system

$$\dot{x}_1(t)=x_2(t-1)$$
$$\dot{x}_2(t)=x_2(t)u(t)$$
$$y(t)=x_1(t)$$

After differentiating, we get

$$\mathrm{d}\dot{x}(t)=A\mathrm{d}x(t)+B\mathrm{d}u(t)$$
$$\mathrm{d}y(t)=C\mathrm{d}x(t)+D\mathrm{d}u(t)$$

where

$$A=\begin{pmatrix}0 & \delta\\ 0 & u(t)\end{pmatrix},\quad B=\begin{pmatrix}0\\ x_2(t)\end{pmatrix},\quad C=\begin{pmatrix}1 & 0\end{pmatrix}$$

Note that

$$(sI-A)^{-1}=\begin{pmatrix}\frac{1}{s} & \frac{\delta}{s^2-u(t-1)s}\\ 0 & \frac{1}{s-u(t)}\end{pmatrix}$$

Finally

$$F(\delta,s)=C(sI-A)^{-1}B=\frac{\delta}{s^2-u(t-1)s}\cdot x_2(t)=\frac{x_2(t-1)\delta}{s^2-u(t-1)s}=\frac{\dot{y}(t)\delta}{s^2-u(t-1)s}$$

In spite of the formal similarity to the transfer functions of linear time-delay systems, inverting matrix $(sI-A)$ over the non-commutative field is now far from trivial, since entries of $(sI-A)$ are skew polynomials from the ring $\mathcal{K}[\delta,s]$. Inversion requires finding a solution to the set of linear equations over a non-commutative field, see [5] and [15].

3 BASIC PROPERTIES OF TRANSFER FUNCTIONS

Transfer functions as defined in previous section have many properties we expect from transfer functions:

- they are invariant with respect to state-transformations,
- they provide input-output description,
- they are related to the accessibility and observability of a nonlinear system,
- they allow us to use the transfer function algebra when combining systems in series, parallel and feedback connection.

3.1 Invariance under state-transformation

Clearly, each linear system has a unique transfer function. Any linear state-space transformation $\xi(t) = Tx(t)$ represents, in fact, a linear transformation under which the transfer function, defined by Laplace transforms, is invariant. However, the same can be said also for transfer functions of nonlinear systems, all of them, continuous-, discrete-time and time-delay. Naturally, one can consider nonlinear state-space transformations $\xi(t) = \phi(x(t))$ as well (respectively, $\xi(t) = \phi(\{x(t-i); i \geq 0\})$ for time-delay case).

Consider, for instance, the discrete-time case (see [3] for more details).

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{aligned} \tag{11}$$

Instead of equations (11), we can equivalently work on differentials

$$\begin{aligned} \mathrm{d}x(t+1) &= A\mathrm{d}x(t) + B\mathrm{d}u(t) \\ \mathrm{d}y(t) &= C\mathrm{d}x(t) + D\mathrm{d}u(t) \end{aligned} \tag{12}$$

where $A = (\partial f(.)/\partial x(t))$, $B = (\partial f(.)/\partial u(t))$, $C = (\partial g(.)/\partial x(t))$ and $D = (\partial g(.)/\partial u(t))$. Equations (12) can be reformulated as

$$\begin{aligned} (\delta I - A)\mathrm{d}x(t) &= B\mathrm{d}u(t) \\ \mathrm{d}y(t) &= C\mathrm{d}x(t) + D\mathrm{d}u(t) \end{aligned} \tag{13}$$

Now, it follows

$$F(\delta) = C(\delta I - A)^{-1} B + D \tag{14}$$

Note again that in spite of the formal similarity to transfer functions of linear discrete-time systems, inverting matrix $(\delta I - A)$ over the non-commutative field is far from trivial, since the entries of $(\delta I - A)$ are skew polynomials.

For any state transformation $\xi(t) = \phi(x(t))$ one has $\mathrm{rank}_{\mathcal{K}} T = n$, where $T = (\partial \phi / \partial x(t))$. Since $\mathrm{d}\xi(t) = T\mathrm{d}x(t)$, in the new coordinates we have

$$\begin{aligned} \mathrm{d}\xi(t+1) &= \delta(T)AT^{-1}\mathrm{d}\xi(t) + \delta(T)B\mathrm{d}u(t) \\ \mathrm{d}y(t) &= CT^{-1}\mathrm{d}\xi(t) + D\mathrm{d}u(t) \end{aligned} \tag{15}$$

where $\delta(T)$ means δ applied pointwise to *T*. Thus, the transfer function reads as

$$F(\delta) = CT^{-1}\left(\delta I - \delta(T)AT^{-1}\right)^{-1} \delta(T)B + D = C\left(\delta\left(T^{-1}\right)\delta \cdot T - A\right)^{-1} B + D \tag{16}$$

After applying the commutation rule $\delta \cdot T = \delta(T) \cdot \delta$, we get $F(\delta) = C(\delta I - A)^{-1} B + D$.

Example 5. Consider the system

$$x_1(t+1)=u(t)$$
$$x_2(t+1)=x_1(t)u(t)$$
$$y(t)=\frac{x_2(t)}{x_1(t)}$$

which is, in fact, linear since $y(t+2)=u(t)$. The state equations can be linearized by the state transformation $\xi_1(t)=x_2(t)/x_1(t)$, $\xi_2(t)=x_1(t)$. Transfer function reads as

$$F(\delta)=\frac{1}{\delta^2}$$

Analogical ideas apply also to nonlinear continuous-time [5] and time-delay systems [7].

3.2 Transfer function algebra

We can also introduce algebra of transfer functions of nonlinear systems. Each system structure can be divided into three basic types of connections: series, parallel and feedback, see Fig. 1.

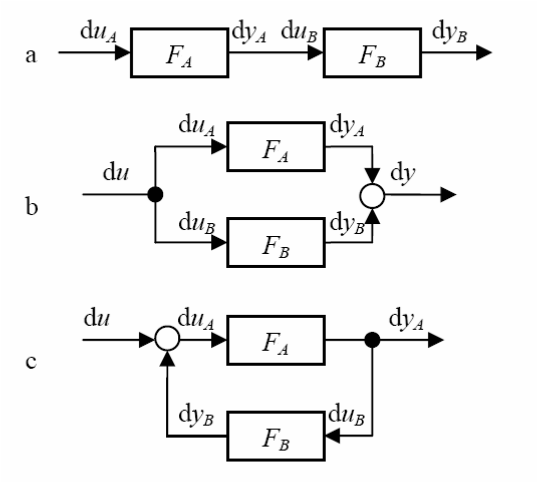

Figure 1: Series, parallel and feedback connection of nonlinear systems

For a series connection it follows that $\mathrm{d}y(t)_B=F_B\mathrm{d}u(t)_B=F_BF_A\mathrm{d}u(t)_A$. Thus

$$F=F_B\cdot F_A \tag{17}$$

For parallel and feedback connection we get

$$F=F_A+F_B \tag{18}$$

$$F=(1-F_A\cdot F_B)^{-1}\cdot F_A \tag{19}$$

Note that due to the non-commutative multiplication (17) and (19) have to be kept exactly as they are.

Following example demonstrates how to handle a series connection of two nonlinear time-delay systems. It also serves as a motivation why to use the transfer function formalism in nonlinear systems.

Example 6. Consider two systems

$$\dot{y}_A(t)=y_A(t)u_A(t-1) \qquad y_B(t)=\ln u_B(t)$$

Transfer functions are following

$$F_A(\delta,s)=\frac{y_A(t)\delta}{s-u_A(t-1)} \qquad\qquad F_B(\delta,s)=\frac{1}{u_B(t)}$$

The systems are combined together in a series connection. For the connection A → B, when $u_B(t)=y_A(t)$, the resulting transfer function is

$$F(\delta,s)=F_B(\delta,s)F_A(\delta,s)=\frac{1}{u_B(t)}\cdot\frac{y_A(t)\delta}{s-u_A(t-1)}=\frac{1}{y_A(t)}\cdot\frac{y_A(t)\delta}{s-u_A(t-1)}$$
$$=\frac{y_A(t)\delta}{y_A(t)s+\dot{y}_A(t)-y_A(t)u_A(t-1)}=\frac{y_A(t)\delta}{y_A(t)s}=\frac{\delta}{s}$$

Hence, the connection A → B is linear from an input-output point of view $\dot{y}_B(t)=u_A(t-1)$. However, when the systems are connected as B → A, that is $u_A(t)=y_B(t)$, the result is different.

$$F(\delta,s)=F_A(\delta,s)F_B(\delta,s)=\frac{y_A(t)\delta}{s-u_A(t-1)}\cdot\frac{1}{u_B(t)}=\frac{y_A(t)\delta}{u_B(t-1)s-u_B(t-1)\ln u_B(t-1)}$$

This time, it does not yield a linear system. Clearly, the fact that the associativity is not valid when systems are nonlinear is hidden in the transfer function algebra which is non-commutative.

Presented example serves also as a stepping stone to the controller design. Naturally, one can employ introduced transfer function algebra to design various types of compensators, for instance to solve the nonlinear model matching problem, as was already done in [9].

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5 CONCLUSIONS

Transfer function formalism in nonlinear control systems is, in principle, similar to the linear theory. There are two main differences: the polynomial description relates now the differentials of system inputs and outputs and the resulting polynomial ring is non-commutative. Such a transfer function formalism has been already employed in [8] to study some structural properties of nonlinear systems, in [9] to study the nonlinear model matching problem, in [17] to study the observer design and recently in [18] to study the realization problem of nonlinear systems.

REFERENCES

[1] CONTE, G., C.H.MOOG, A.M.PERDON: Nonlinear control systems: an algebraic setting. Springer-Verlag, London, 1999.

[2] HALÁS, M., M.HUBA: Symbolic computation for nonlinear systems using quotients over skew polynomial ring. In 14th Mediterranean Conference on Control and Automation, Ancona, Italy, 2006.

[3] HALÁS, M., Ü.KOTTA: Extension of the concept of transfer function to discrete-time nonlinear control systems. In European Control Conference, Kos, Greece, 2007.

[4] ARANDA-BRICAIRE, E., Ü.KOTTA, C.H.MOOG: Linearization of discrete-time systems. SIAM Journal of Control Optimization, **34**, pp. 1999-2023, 1996.

[5] HALÁS, M.: An algebraic framework generalizing the concept of transfer functions to nonlinear systems. Automatica, **44**, scheduled for the May, 2008.

[6] XIA, X., L.A.MÁRQUEZ-MARTÍNEZ, P.ZAGALAK, C.H.MOOG: Analysis of nonlinear time-delay systems using modules over non-commutative rings. Automatica, **38**, pp. 1549-1555, 2002.

[7] HALÁS, M.: Nonlinear time-delay systems: a polynomial approach using Ore algebras. In Loiseau, J.J., Michiels, W., Niculescu, S., Sipahi, R.: Topics in Time-Delay Systems: Analysis, Algorithms and Control. Lecture Notes in Control and Information Sciences. Springer, Berlin/Heildelberg, pp. 109-119, 2009.

[8] PERDON, A.M., C.H.MOOG, G.CONTE: The pole-zero structure of nonlinear control systems. In 7th IFAC Symposium NOLCOS, Pretoria, South Africa, 2007.

[9] HALÁS, M., Ü.KOTTA, C.H.MOOG: Transfer function approach to the model matching problem of nonlinear systems. In 17th IFAC World Congress, Seoul, Korea, 2008.

[10] HALÁS, M., Ü.KOTTA: Transfer Functions of Discrete-time Nonlinear Control Systems. Estonian Acad. Sci. Phys. Math., **56**, pp. 322–335, 2007.

[11] JOHNSON, J.: Kähler differentials and differential algebra. Annals of Mathemathics, **89**, pp. 559–571, 1976.

[12] FLIESS, M., LÉVINE, J., MARTIN, P., ROUCHON, P.: Flatness and defect of non-linear systems: introductory theory and examples. Int. J. Control, **61**, pp. 1327–1361, 1995.

[13] ZHENG, Y., WILLEMS, J., ZHANG, C.: A polynomial approach to nonlinear system controllability. IEEE Transactions on Automatic Control, **46**, pp. 1782–1788, 2001.

[14] ZHENG, Y., CAO, L.: Transfer function description for nonlinear systems. Journal of East China Normal University (Natural Science), **2**, pp. 15–26, 1995.

[15] ORE, O.: Linear equations in non-commutative fields. Annals of Mathemathics, **32**, pp. 463–477, 1931.

[16] ORE, O.: Theory of non-commutative polynomials. Annals of Mathemathics, **34**, pp. 480–508, 1933.

[17] HALÁS, M., Ü.KOTTA: A polynomial approach to the synthesis of observers for nonlinear systems. In 47th IEEE Conference on Decision and Control, Cancun, Mexico, 2008.

[18] HALÁS, M., Ü.KOTTA: Realization problem of SISO nonlinear systems: a transfer function approach. In 7th IEEE Conference on Control & Automation, Christchurch, New Zealand, 2009.